Kramers-Kronig relations with logarithmic kernel and application to the phase spectrum in the Drude model

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Standard Kramers-Kronig relations are formulated on the premise that the response functions are well behaved asymptotically. In certain physical problems in which the functions are logarithmic, one may then need to reformulate these relations. This was recently pointed out very explicitly in an optical context by Nash, Bell, and Alexander [J. Mod. Opt. **42**, 1837 (1995)]. Much earlier this issue was discussed more generally. We examine in some detail the mathematical problem by considering the phase spectrum in the Drude model. Comparison is made between the standard and the reformulated forms of Kramers-Kronig relations. [S1063-651X(97)01310-X]

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I. INTRODUCTION

As is well known, Kramers-Kronig relations connect the real and imaginary parts of a response function such as the dynamic susceptibility via integral transforms. These relations are very general as they do not depend on properties of models (other than perhaps the Hermitianness implicitly). They are now among standard tools of analysis in dynamical theory of many-particle systems [1]. They lead to a number of useful sum rules, e.g., the susceptibility sum rule by which the validity of approximate theories may be assessed. For examples of recent applications, see Miller and Richards in high T_c superconductivity [2], Peiponen and Vartianen in optics [3], Gorges, Grosse, and Theiss in dielectric functions for mixtures [4], Tan and Callaway in conductivities for strongly correlated electrons and Sturm in optical sum rules for inhomogeneous electron systems [5], and our work in the relaxation theory of a semiclassical gas [6].

Usually Kramers-Kronig (KK) relations are derived with an assumption that the response functions are well behaved asymptotically. That is, they vanish ''sufficiently fast'' as the frequency becomes very large. The susceptibility for an electron gas, for example, satisfies this condition very easily as do other similar response functions [7]. If these functions do not vanish ''sufficiently fast,'' it would appear that KK relations must be reformulated. There may indeed be physical problems for which this reformulation becomes necessary.

Recently, Nash, Bell, and Alexander [8] in an interesting work drew attention to such a possibility. If light impinges normally on the surface of a metal, the complex reflection coefficient r may be expressed as $r=\rho\ \exp i\Psi$, where ρ is the amplitude and Ψ the phase. The dispersion relation for the phase Ψ would then encounter $\ln\rho$ and this log function may not behave well. By reformulating KK relations they have deduced the following expression:

$$\Psi(\omega)=\Psi(0)-\frac{2\omega}{\pi}\,\mathrm{P}\int_0^\infty\frac{\ln\rho(u)\,du}{(u^2-\omega^2)},\tag{1}$$

where P means a principal-value (PV) integral and $\rho(\omega)$ is assumed even. This expression differs from the usual one only in the constant, the first term on the right-hand side of Eq. (1).

Following their analysis, it is not difficult to write down a general formula for a function f, where $f(x)/x$ is assumed to behave well when $|x|\to\infty$:

$$f(x)=f(0)+\frac{x}{\pi i}\,\mathrm{P}\int_{-\infty}^\infty\frac{f(y)dy}{y(y-x)}.\tag{2}$$

If one writes $f=f_1+if_2$,

$$f_1(x)=f_1(0)+\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^\infty\frac{f_2(y)dy}{y(y-x)},\tag{2a}$$

and

$$f_2(x)=f_2(0)-\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^\infty\frac{f_1(y)dy}{y(y-x)}.\tag{2b}$$

Equation (1) corresponds to taking $f_2=\Psi$ with an even f_1 in Eq. (2b). Observe that f_2 may not be an odd function unless $f_2(0)=0$, but f_1 may be an even function. See Appendix A for an alternative derivation of Eq. (2).

Nash, Bell, and Alexander do not make use of their new relation (1) to evaluate the phase directly. That would involve evaluating the PV log integral itself. Instead they use well-known optical relationships to show that for the Drude model there can indeed exist a nonzero constant term, which would be absent according to the usual form of KK relations. Their demonstration of the existence of this constant term lends credence to their original premise.

In this work we shall show that the PV integral can be evaluated, recovering the result of Nash, Bell, and Alexander. This work shows a new approach to solving these PV log integrals directly. In standard methods one evaluates this kind of integral indirectly by using Cauchy's theorem. Thus our alternative method may be of some interest and perhaps even prove advantageous by being a direct approach.

Before doing so, we should briefly point out some relevant history pertaining to this special situation. As is undoubtedly well known, the dispersion relation credited to Kramers and Kronig originated in optical scattering problems. In the 1950s there were already suggestions made that the boundedness condition that had been required in formulating the dispersion relation might be too limiting. These considerations have been brought up in the context of the scattering matrix theory of elementary particles. It would appear that van Kampen [9] might have been the first to consider explicitly a response function which behaves well only in the form that we have given, i.e., $f(x)/x$ as $|x|\to\infty$. He obtained a formula essentially identical to Eq. (2). Somewhat earlier Rohrlich and Gluckstern [10] gave a dispersion relation for a function which includes $f(x)$ given above. A little later Toll [11] also indicated the existence of a constant term. There is a review given by Stern [12] on these early developments of the dispersion relations where one can glean some aspects of this particular issue. Until Nash, Bell, and Alexander, no one, to our knowledge, studied the dispersion relations with log functions explicitly. Please see the note added in proof.

II. DRUDE MODEL

The Drude model is perhaps the simplest model of metals, accounting for some of the basic properties of what may be termed a free-electron metal [13]. If light impinges normally on its surface, the amplitude of the reflectivity for the Drude model is given as follows [8,13]: If ω_p is the classical plasma frequency,

$$\rho(\omega)=\begin{cases}1, & 0<\omega<\omega_p\\ \dfrac{\omega-\sqrt{\omega^2-\omega_p^2}}{\omega+\sqrt{\omega^2-\omega_p^2}}, & \omega>\omega_p\,.\end{cases}\tag{3}$$

Observe that $\rho(\omega\to\infty)=(\omega_p/\omega)^2/4$. Hence at large frequencies, $|\ln\rho|$ increases logarithmically with ω.

Now Eq. (1) may be slightly rewritten as follows:

$$\Psi(\omega)-\Psi(0)=-\frac{1}{\pi}\,\mathrm{P}\int_0^\infty \ln\left|\frac{u+\omega}{u-\omega}\right|\,\frac{d\,\ln\rho}{du}\,du.\tag{4}$$

Using the Drude model form for ρ given above, we obtain

$$\Psi(\omega)-\Psi(0)=\frac{2}{\pi}\,\mathrm{P}\int_{\omega_p}^\infty \ln\left|\frac{u+\omega}{u-\omega}\right|\,\frac{du}{\sqrt{u^2-\omega_p^2}}$$

$$\to\frac{2}{\pi}\,\mathrm{P}\int_1^\infty \ln\left|\frac{u+\omega}{u-\omega}\right|\,\frac{du}{\sqrt{u^2-1}}\equiv I,\tag{5}$$

where now u and ω are dimensionless. To simplify the above integral, let $u=1/x$ and also $\omega=1/a$, where a is to be regarded as a fixed number, i.e., $a>1$ (low frequency) and $0<a<1$ (high frequency). Then,

$$I(a)=\frac{2}{\pi}\,\mathrm{P}\int_0^1 \ln\left|\frac{a+x}{a-x}\right|\,\frac{dx}{x\sqrt{1-x^2}}.\tag{6}$$

To eliminate the square root, we introduce two simple changes apparently due to Watson [14]: First, let $x=\cos\theta$ and second, $\tan\theta/2=y$. Then,

$$\overline{I(a)}\equiv I(a)\Big/\left(\frac{2}{\pi}\right)=\mathrm{P}\int_0^\infty \ln\left|\frac{c^2+y^2}{1+c^2y^2}\right|\,\frac{dy}{1-y^2},\tag{7}$$

where $c^2=(a+1)/(a-1)$, i.e., $c^2>1$ if $a>1$ (low frequency): $c^2<0$ if $0<a<1$ (high frequency). We shall treat the two different cases separately, but in a parallel manner to show whence the different behavior originates.

A. Low-frequency domain

The first case is simpler since $I(a)$ is no longer a PV integral but an ordinary integral and also the absolute sign becomes unnecessary. But we will retain the PV sign to be able to separate the log term in Eq. (7). We shall treat this low-frequency case first. Then,

$$\overline{I(a)}=\mathrm{P}\int_0^\infty \ln(c^2+y^2)\,\frac{dy}{1-y^2}-\mathrm{P}\int_0^\infty \ln(1+c^2y^2)\,\frac{dy}{1-y^2}.\tag{8}$$

By letting $y\to 1/y$ in the second integral of Eq. (8), one may write it as

$$\overline{I(a)}=\pi^2/2+2\,\mathrm{P}\int_0^\infty \ln(c^2+y^2)\,\frac{dy}{1-y^2}$$

$$=\pi(2\arctan c-\pi/2)=\pi\arcsin(1/a).\tag{9}$$

See Eq. (13) and Appendix B for an evaluation of the above log integral. For $a>1$ (i.e., $c^2>0$), Eq. (6) may also be solved by differentiation noting that $I(a=\infty)=0$. Hence

$$I(a)=2\arcsin(1/a),\quad a>1.\tag{10}$$

B. High-frequency domain

Now we shall turn to the second case where $0<a<1$, i.e., $c^2<0$. Let $b^2=-c^2>0$. As before, we can split Eq. (7) into two terms, each of which is also well defined being a PV integral,

$$\overline{I(a)}=\mathrm{P}\int_0^\infty \ln|b^2-y^2|\,\frac{dy}{1-y^2}-\mathrm{P}\int_0^\infty \ln|1-b^2y^2|\,\frac{dy}{1-y^2}.\tag{11}$$

Let $y\to 1/y$ in the second integral. Then,

$$\overline{I(a)}=-2\int_0^\infty \ln y\,\frac{dy}{1-y^2}+\mathrm{P}\int_{-\infty}^\infty \ln|b^2-y^2|\,\frac{dy}{1-y^2}$$

$$\equiv -2A+B.\tag{12}$$

We shall evaluate the two integrals A and B separately. Observe that A is an ordinary integral, also occurring in Eq. (9). It can be written as follows:

$$A=\int_0^1 \ln y\left\{\frac{1}{1-y}+\frac{1}{1+y}\right\}dy=-\mathrm{Li}_2(1)+\mathrm{Li}_2(-1)$$
$$=-\pi^2/4, \tag{13}$$

where Li_2 denotes the dilog of Euler [15,16]. See Appendix C. Now turning to the remaining integral, which is a PV integral, we find that $B=0$, proved in Appendix D. Putting together our solutions for A and B, we obtain

$$\overline{I(a)}=\pi^2/2. \tag{14}$$

Now combining both cases,

$$I(a)=\begin{cases}\pi, & 0<a<1\\ 2\arcsin(1/a), & a>1.\end{cases} \tag{15}$$

Finally, restoring the frequency unit,

$$\Psi(\omega)=\begin{cases}\Psi(0)+\pi=0 \quad \text{if } \omega>\omega_p & \text{(16a)}\\ \Psi(0)+2\arcsin(\omega/\omega_p)=-\pi+2\arcsin(\omega/\omega_p) \quad \text{if } 0<\omega<\omega_p. & \text{(16b)}\end{cases}$$

This value $\Psi(0)=-\pi$ can be readily determined by the boundary condition that $\Psi(\omega\to\infty)=0$ [13]. This recovers the results obtained by Nash, Bell, and Alexander [8]. When $\omega=0$, the reflection is total. Hence the phase is reversed. When $\omega=\infty$, there is a total transmission. Hence the phase is unchanged. For the Drude model the total transmission of course takes place if $\omega>\omega_p$ [13].

III. CONCLUDING REMARKS

One can determine where the different behavior in the high and low frequencies comes from by examining the two essential integrals involved. From Eqs. (7) and (11),

$$I(a>1)=\mathrm{P}\int_0^\infty \ln\left(\frac{c^2+x^2}{1+c^2x^2}\right)\frac{dx}{1-x^2}$$
$$=\pi^2/2+2\mathrm{P}\int_0^\infty \ln(x^2+c^2)\,\frac{dx}{1-x^2}, \tag{17}$$

where $c^2=(a+1)/(a-1)>1$, and

$$I(0<a<1)$$
$$=\mathrm{P}\int_0^\infty \ln\left|\frac{b^2-x^2}{1-b^2x^2}\right|\frac{dx}{1-x^2}$$
$$=\pi^2/2+2P\int_0^\infty \ln|b^2-x^2|\,\frac{dx}{1-x^2}, \tag{18}$$

where $b^2=(1+a)/(1-a)>1$, $b^2=-c^2$. Thus the different behavior in the high and low frequencies must come from the log arguments in the right-hand side of Eqs. (17) and (18). The PV log integral in Eq. (17) is finite and remains parameter-a dependent. But the PV log integral in Eq. (18) vanishes, losing the parameter-a dependence altogether.

It should be remembered that when $a>1$, $I(a)$, as pointed out earlier, really is not a PV integral, but an ordinary integral. Such an integral is expected to retain the parameter dependence. When $a<1$, $I(a)$ is truly a PV integral, which as a rule loses the parameter dependence [17,18]. This difference can be shown in another way: The roots of the log arguments of Eqs. (17) and (18) are imaginary and real, respectively. By this property one can transform the paths of integration so that they do not follow along the real axis for Eq. (17) but they do for Eq. (18). For the latter case, Eq. (18), the parameter which appears as a scale factor in the log argument can be scaled out since the paths of integration follow the real axis. For the former case, Eq. (17), this is not possible.

It is now interesting to ask whether the general expression (2) reduces to the standard one if, as $|x|\to\infty$, $f(x)$ were made to behave well. Let us denote such an f by g, i.e., $g(x)\to 0$ as $|x|\to\infty$. Clearly $g(x)/x\to 0$ also in this limit. Hence Eq. (2) is applicable to g. For this class of functions we know [1] that

$$g(x)=\frac{1}{\pi i}\,\mathrm{P}\int_{-\infty}^{\infty} g(y)\,\frac{dy}{y-x}. \tag{19}$$

Hence, by writing $g=g_1+ig_2$,

$$g_1(x)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} g_2(y)\,\frac{dy}{y-x} \tag{19a}$$

and

$$g_2(x)=-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} g_1(y)\,\frac{dy}{y-x}. \tag{19b}$$

These are all well known and we have referred to them as the standard expressions.

Now, by replacing f by g in Eq. (2a)

$$g_1(x)=g_1(0)+\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} g_2(y)\,\frac{dy}{y(y-x)}. \tag{20}$$

As for $g_1(0)$, we can obtain it from Eq. (19a) by setting $x=0$,

$$g_1(0)=\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} g_2(y)\,\frac{dy}{y}, \tag{21}$$

which may be recognized as the static susceptibility sum rule if g is the magnetic response function. If one substitutes Eq. (21) for the first term on the right-hand side of Eq. (20), we at once recover Eq. (19a), the standard expression.

Also replacing f by g now in Eq. (2b),

$$g_2(x)=g_2(0)-\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}g_1(y)\,\frac{dy}{y(y-x)}$$
$$=g_2(0)-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}g_1(y)\,\frac{dy}{y-x}, \qquad (22)$$

where we have assumed that g_1 is an even function. If we let $|x|\to\infty$,

$$g_2(x\to\infty)=g_2(0)+\frac{1}{\pi x}\,\mathrm{P}\int_{-\infty}^{\infty}g_1(y)dy$$
$$=g_2(0)+g_2(x\to\infty), \qquad (23)$$

where the second term on the right-hand side of Eq. (23) follows from Eq. (19b), the standard expression. By comparing both sides of Eq. (23), we see that $g_2(0)=0$. Hence Eq. (22) is the same as Eq. (19b). Thus if f behaves well, Eqs. (2a) and (2b) reduce to Eqs. (19a) and (19b), respectively.

If $f(x)$ behaves well, there is no constant term in the imaginary part of $f=g$. One can see at once from Eq. (19b) that if $x=0$,

$$g_2(x=0)=-\frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}g_1(y)\,\frac{dy}{y}. \qquad (24)$$

It vanishes since g_1 is an even function. But if f behaves well only as $f(x)/x$, $f_2(x=0)$ does not have this form and is generally nonvanishing. In fact, as $|x|\to 0$,

$$f_2(x)=f_2(0)-\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}f_1(y)\,\frac{dy}{y^2}, \qquad (25)$$

but

$$g_2(x)=-\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}g_1(y)\,\frac{dy}{y^2}. \qquad (26)$$

Thus the constant term $f_2(0)$ exists if the function f (such as a log function) is asymptotically well behaved in the form $f(x)/x$ but not in itself otherwise. This constant is thus a telltale sign, which appears in the phase spectrum in the Drude model. Our conclusion complements the physical discussion on the origin of the constant term by Nash, Bell, and Alexander.

Note added in proof. It would appear that F. C. Jahoda [Phys. Rev. **107**, 1261 (1957)] was the first to express the reflectivity by an amplitude and a phase to use them as complementary variables in a *KK* analysis of barium oxide. Thereafter H. R. Philipp and E. A. Taft [Phys. Rev. **113**, 1002 (1959)] and H. R. Philipp and H. Ehrenreich [Phys. Rev. **129**, 1550 (1965)] have developed and extensively applied this idea to Ge and others. In these experimental works the complex index of refraction is $N=n-i\kappa$, κ being the extinction coefficient, whereas in the work of Nash *et al.*, $N=n+i\kappa$ (see also p. 462, Ref. [1]), resulting in a sign difference in the phase. Also note that for Ge, $\Psi(\omega=0)=0$ since $\kappa(\omega=0)=0$. Hence the general existence of this constant could not have been detected by their experiments. We thank Professor G. D. Mahan for drawing our attention to the existence of these earlier works and Dr. H. R. Philipp for an informative discussion on the first applications of the *KK* analysis to optical properties of solids.

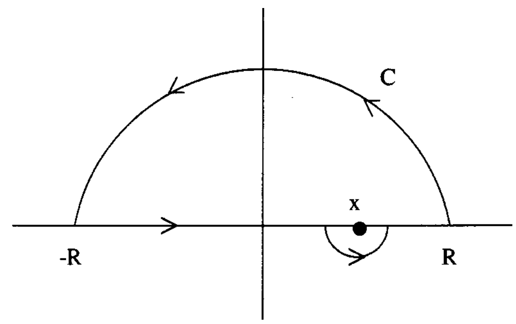

FIG. 1. The path of integration C for Eq. (A1) indicated by arrows in the upper half plane. Here R is assumed very large. The indented path about x forms a very small semicircle. The path C encloses the singular point at x.

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We thank Dr. M. Lax for informing us of several early references on Kramers-Kronig relations (Refs. [9–12]). They have been very useful for our work. One of us (M.H.L.) thanks Dr. E. B. Brown and Dr. P. L. Nash for discussions on the work of Ref. [8], and Dr. K. T. R. Davies for correspondence on PV integrals. This work has been supported in part by NATO (Grant No. CRG 921268).

APPENDIX A: KRAMERS-KRONIG RELATIONS BY ROHRLICH AND GLUCKSTERN

We shall reproduce here Kramers-Kronig relations given earlier by Rohrlich and Gluckstern [10], slightly modified for our problem on hand. Since this derivation depends only on a general property of functions, it can be compared with the derivation due to Nash, Bell, and Alexander [8]. In any event one may regard it as an alternative to the work of Nash, Bell, and Alexander.

By Cauchy's theorem for an analytic function

$$W(z)=\frac{1}{2\pi i}\int_c W(s)\,\frac{ds}{s-z} \qquad (A1)$$

for a closed contour C. Now suppose that z takes on real values only, and also assume that the function W is analytic in the upper half plane (a critical assumption). We can then choose the path C of integration in the plane of s to follow along the real axis of s from $-R$ to R but indented around $z=x$ with a small arc in the positive direction and finally along a large arc of radius R in the upper half plane of s to complete a circuit. See Fig. 1. Then,

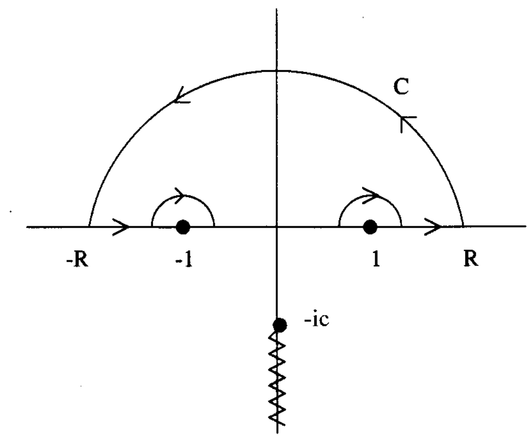

FIG. 2. The path of integration C for Eq. (B1) indicated by arrows in the upper half plane. Here R is assumed very large. The indented paths about -1 and 1 form very small semicircles. The path C encloses no singularities. The wiggly line from $-ic$ to $-i\infty$, $c>0$, denotes a branch line of the log term in Eq. (B1). The path from $-R$ to R may also be deformed to make a great semicircle in the lower half plane which excludes the branch line and the poles at $-$ and 1.

$$W(x)=\frac{1}{2\pi i}\,\mathrm{P}\int_{-R}^{R}W(s)\,\frac{ds}{s-x}+\frac{1}{2}\,W(x)$$
$$+\frac{1}{2\pi i}\int_{\mathrm{arc}\;R}W(s)\,\frac{ds}{s-x}. \tag{A2}$$

If $R\to\infty$, one may let $s-x\to s$ in the third term on the right-hand side of Eq. (A2). Now since $W(s)$ is analytic in the upper half plane, the path of integration along the arc of radius R may be deformed to coincide with the real axis, going from R to $-R$ except at $s=0$, where it is indented in the positive direction. Hence

$$W(x)=W(0)+\frac{1}{\pi i}\,\mathrm{P}\int_{-\infty}^{\infty}W(s)\,\frac{ds}{s-x}+\frac{1}{\pi i}\,\mathrm{P}\int_{\infty}^{-\infty}W(s)\,\frac{ds}{s}$$
$$=W(0)+\frac{x}{\pi i}\,\mathrm{P}\int_{-\infty}^{\infty}W(s)\,\frac{ds}{s(s-x)}. \tag{A3}$$

This is the result given by Rohrlich and Gluckstern except for the contribution at the origin $W(0)$. This term was absent as they have assumed that $W(s)/s$ is regular at $s=0$. By writing $W=W_1+iW_2$, we obtain at once

$$W_1(x)=W_1(0)+\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}W_2(s)\,\frac{ds}{s(s-x)} \tag{A4}$$

and

$$W_2(x)=W_2(0)-\frac{x}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty}W_1(s)\,\frac{ds}{s(s-x)}. \tag{A5}$$

Equation (A3) corresponds to Eq. (2). If we take $W_1=f_1=\ln\rho$ and $W_2=f_2=\Psi$, Eqs. (1) and (A5) are identical provided of course that $W_1(x)$ is an even function of x. It is interesting to compare the assumptions on f and W attached. Nash, Bell, and Alexander require that their function f be such that $f(x)/x\to 0$ sufficiently fast as $|x|\to\infty$. Rohrlich and Gluckstern require that their function W be regular in the upper half plane.

APPENDIX B

$$\mathrm{P}\int_0^{\infty}\ln(x^2+c^2)[dx/(1-x^2)]=-\pi\,\arctan(1/c),\quad c>0.$$

To prove the above, let us define an integral M by

$$M=\int_c \ln(z+ic)\,\frac{dz}{1-z^2},\quad c>0 \tag{B1}$$

where the contour C denotes a closed path along the real axis indented at $z=-1$ and $z=+1$ and a semicircle enclosing the upper half of the z plane. See Fig. 2. There is a branch cut running from $z=-ic$, $c>0$, to $z=-i\infty$, which is thus outside the contour C. Now $M=0$ since the contour does not enclose any singularities. M is contributed by integrals along the small indented paths about $z=-1$ and 1, and the remainder of the real axis, which forms a PV integral from $-\infty$ to ∞, i.e., $M=M_{-1}+M_1+M_p$ in obvious notation. They may be separately evaluated:

$$M_{-1}+M_1=(\pi i/2)\ln\left(\frac{1+ic}{1-ic}\right)+\pi^2/2=-\pi[\arctan c-\pi/2]$$
$$=\pi\,\arctan(1/c),\quad c>0 \tag{B2}$$

$$M_p=\mathrm{P}\int_{-\infty}^{\infty}\ln(x+ic)\,\frac{dx}{1-x^2}=\mathrm{P}\int_0^{\infty}\ln(x^2+c^2)\,\frac{dx}{1-x^2}. \tag{B3}$$

Hence we obtain

$$\mathrm{P}\int_0^{\infty}\ln(x^2+c^2)\,\frac{dx}{1-x^2}=-\pi\,\arctan(1/c),\quad c>0.\quad \text{Q.E.D.} \tag{B4}$$

The validity of the solution can be tested by comparing the expansions of both sides in powers of $1/c$, i.e., c large. To do so, one may not simply expand the log term on the left-hand side of Eq. (B4). Instead let $x\to cx$ in the PV integral in Eq. (B4), which breaks the PV integral into two terms. The one containing $\ln c^2$, however, vanishes. The other can now be expanded simply in powers of $1/c$, to yield the expansion of the right-hand side of Eq. (B4). One may also differentiate both sides of Eq. (B4) with respect to c. The resulting PV integral is elementary to evaluate since it is no longer a log integral.

APPENDIX C: DILOG OF EULER

The dilog of Euler is a transcendental function, denoted Li_2 [15]. It may be defined by the following integral representation:

$$\mathrm{Li}_2(z) = -\int_0^z \ln(1-z)\,\frac{dz}{z}. \tag{C1}$$

This function is complex if $z=x>1$, but real if $-\infty<x<1$. When $z=x=1$ or -1 (also for a few others), the dilog has a special value. For example, $\mathrm{Li}_2(1)=\pi^2/6$, $\mathrm{Li}_2(-1)=-\pi^2/12$. There are also a family of functional relations due to Euler, e.g., duplication, inversion, reflection relations. After the dilog, there is the trilog of Landen, denoted by Li_3. For recent physical applications, see Ref. [16].

APPENDIX D: PROOF THAT $B=0$

The integral B, defined in Eq. (12), is a PV integral:

$$\begin{aligned} B &= \mathrm{P}\int_{-\infty}^{\infty} \ln|b^2-y^2|\,\frac{dy}{1-y^2} \\ &= \mathrm{P}\int_{-\infty}^{\infty} \ln|b+y|\left\{\frac{1}{1-y}+\frac{1}{1+y}\right\}dy, \end{aligned} \tag{D1}$$

where we assume without loss of generality that $b>+1$. Let $y\to y-b$. Then let $y\to py$ and $y\to qy$, respectively, in the first and second terms of Eq. (D1), where $p=b+1$ and $q=b-1$, i.e., $p,q>0$. Then,

$$B=\left(\ln\frac{p}{q}\right)\,\mathrm{P}\int_{-\infty}^{\infty}\frac{dy}{1-y}=0. \tag{D2}$$

One can give more rigorous proof for this type of problem. See Appendix A of Ref. [17].

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